INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 5.18-5.19]

DIFFERENTIAL EQUATIONS

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A. Paper 1 questions (SHORT)

D.E. OF SEPARABLE VARIABLES

1. [Maximum mark: 7]

Solve de differential equations

(a) $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x + 5$ with $y(0) = 3$. [3]

(b) $\frac{\mathrm{d}y}{\mathrm{d}x} = \sin x + \cos x$ with $y(0) = 3$. [4]

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2. [Maximum mark: 5]

Given that $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{e}^x - 2x$ and $y = 3$ when $x = 0$, find an expression for y in terms of x.

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3. [Maximum mark: 5]

Solve de differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x}{3y^2}$ with $y(0) = 2$.

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4. [Maximum mark: 13]

Solve de differential equations, by expressing your solution in the form $y = f(x)$

(a) $\frac{dy}{dx} = 10x^4$ with $y(1) = 2$. [3]

(b) $\frac{dy}{dx} = 10x^4 y^2$ with $y(1) = 2$. [5]

(c) $\frac{dy}{dx} = 10x^4 y$ with $y(1) = 2$. [5]

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5. [Maximum mark: 5]

Solve the differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} = 2xy^2$ given that $y = 1$ when $x = 0$. Give your answer in the form $y = f(x)$.

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6. [Maximum mark: 6]

Solve the differential equation $xy\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + y^2$, given that $y = 0$ when $x = 2$.

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7. [Maximum mark: 7]

Solve the differential equation $(x^2+1)\frac{dy}{dx}-xy=0$ where $x>0,\ y>0$, given that $y=1$ when $x=1$. Give your answer in the form $y=\sqrt{\frac{x^2+a}{b}}$

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8. [Maximum mark: 7]

(a) Show that $\dfrac{1}{y-y^2}=\dfrac{1}{y}+\dfrac{1}{1-y}$

(b) **Hence**, show that the general solution of the differential equation $\dfrac{\mathrm{d}y}{\mathrm{d}t}=k(y-y^2)$, $0<y<1$, is $y=\dfrac{1}{1+Ae^{-kt}}$ (where A is a constant)

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9. [Maximum mark: 6]

The equation of motion of a particle with mass m, subjected to a force kx can be written as $kx = mv\frac{\mathrm{d}v}{\mathrm{d}x}$, where x is the displacement and v is the velocity.

When $x = 0$, $v = v_0$. Find v, in terms of m, k and v_0.

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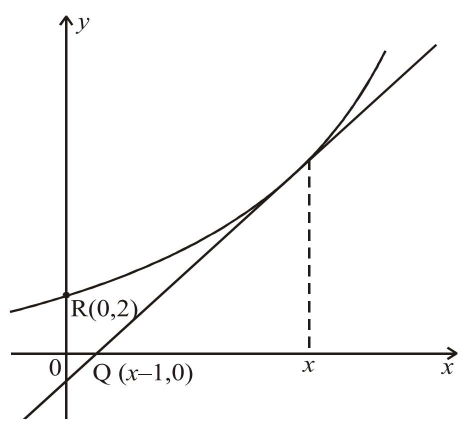
10. [Maximum mark: 6]

The tangent to the curve $y = f(x)$ at the point $P(x, y)$ meets the x-axis at $Q(x-1, 0)$.

The curve meets the y-axis at $R(0, 2)$.



(a) Write down a differential equation that represents this information.

(b) **Hence**, find the equation of the curve.

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11. [Maximum mark: 9]

When air is released from an inflated balloon it is found that the rate of decrease of the volume of the balloon is proportional to the volume of the balloon. This can be represented by the differential equation $\frac{\mathrm{d}v}{\mathrm{d}t} = -ky$, where v is the volume, t is the time and k is the constant of proportionality.

(a) If the initial volume of the balloon is v_0, find an expression, in terms of k, for the volume of the balloon at time t. [4]

(b) Find an expression, in terms of k, for the time when the volume is $\frac{v_0}{2}$. [2]

It is given that that the initial volume of a balloon is 3.6 units^3 and the half-life time of its volume is 5 hours.

(c) Find an expression for the volume of this balloon at time t. [2]

(d) Find the volume of this balloon after 24 hours [1[

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12. [Maximum mark: 8]

A sample of radioactive material M decays at a rate which is proportional to the amount of material present in the sample.

(a) Construct a differential equation for M [1]

(b) Show that M has the form $M = Ae^{kt}$. [3]

(c) Find the half-life of the material if 50 grams decay to 48 grams in 10 years. [4]

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EULER'S METHOD

13. [Maximum mark: 5]

Consider the differential equation $\frac{dy}{dx} = x^2 + y^2$ where $y = 1$ when $x = 0$.

(a) Use Euler's method with $h = 0.1$ to find an approximate value of y when $x = 0.4$. [4]

(b) Write down, giving a reason, whether your approximate value for y is greater than or less than the actual value of y. [1]

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14. [Maximum mark: 5]

Given that $\frac{dy}{dx} - 2y^2 = e^x$ and $y = 1$ when $x = 0$, use Euler's method with step length 0.1 to find an approximate value of y when $x = 0.4$. Give all intermediate values with maximum possible accuracy.

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15. [Maximum mark: 5]

Consider the differential equation $\frac{dy}{dx} = \frac{y^2 + x^2}{2x^2}$ for which $y = -1$ when $x = 1$.

Use Euler's method with a step length of 0.25 to find an estimate for the value of y when $x = 2$.

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16. [Maximum mark: 5]

A curve that passes through the point (1, 2) is defined by the differential equation

$$\frac{dy}{dx} = 2x\left(1 + x^2 - y\right).$$

(a) Use Euler's method to get an approximate value of y when $x = 1.3$, taking steps of 0.1. Show intermediate steps to four decimal places in a table.

(b) How can a more accurate answer be obtained using Euler's method?

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17. [Maximum mark: 7]

Consider the differential equation $\frac{\mathrm{d}y}{\mathrm{d}y} + \frac{xy}{4-x^2} = 1$, where $|x| < 2$ and $y = 1$ when $x = 0$.

(a) Use Euler's method with $h = 0.25$, to find an approximate value of y when $x = 1$, giving your answer to two decimal places. [4]

The exact solution of this equation is $y = \sqrt{4-x^2}\left(\sin^{-1}(0.5x) + 0.5\right)$

(b) Show that your approximate value of y in (a) is greater than the true value of y. [1]

(c) Sketch the graph of $y = f(x)$ for $0 \le x \le 1$. Use your sketch to explain (b). [2]

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SLOPE FIELD

18. [Maximum mark: 8]

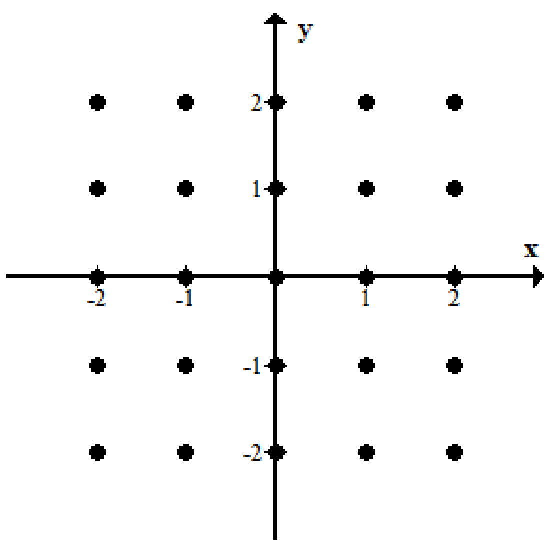
Let $\frac{\mathrm{d}y}{\mathrm{d}x} = x + y$

(a) Complete the table with the values of $\frac{\mathrm{d}y}{\mathrm{d}x}$ for several values of x and y.

	2					
	1					
y	**0**					
	-1					
	-2					
$\frac{\mathrm{d}y}{\mathrm{d}x}$		**-2**	**-1**	**0**	**1**	**2**
		x				

[3]

(b) Hence complete the slope field below (indicate the slope at each point)



[3]

(c) On your diagram above, sketch the particular solution that passes through the point (1,0). [2]

B. Paper 2 questions (LONG)

19. [Maximum mark: 22]

The equation $x^2 + y^2 = 1$ represents the unit circle with its centre at the origin.

The circle passes through the point (0,1).

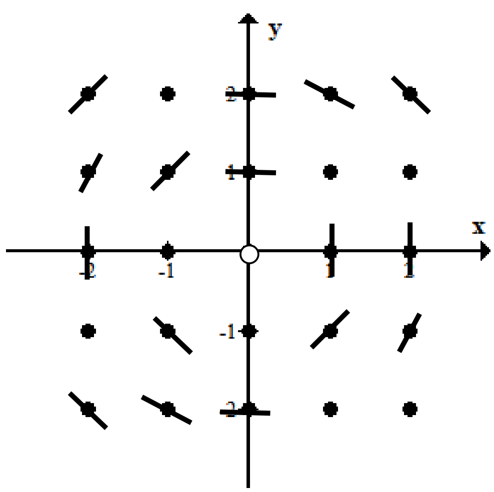
(a) Solve $x^2 + y^2 = 1$ for y to obtain two functions and show that they both satisfy the differential equation $\frac{dy}{dx} = -\frac{x}{y}$. [7]

(b) Solve the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ given that the curve passes through (0,1) to derive the equation of the unit circle. [5]

(c) For $x = \frac{1}{2}$, the true positive value of y is $y = \frac{\sqrt{3}}{2} \cong 0.866$.

Use Euler's method with $h = 0.01$ to find an approximate value of y for $x = 0.5$ and find the percentage error of your estimation. [5]

(d) Part of the slope field of the differential equation $\frac{dy}{dx} = -\frac{x}{y}$ is shown below



(i) Complete the **eight** slopes missing on the slope field (except the point (0,0)). [3]

(ii) Sketch on the same diagram the particular solution passing through (0,1). [2]

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20. [Maximum mark: 7]

Let $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{y}{x}$, with $y(1) = 2$.

(a) Solve the differential equation. [4]

(b) Find the actual value of $y(2)$. [1]

(c) Use Euler's method to find an approximation for $y(2)$ as follows

(i) use a step of $h = 0.2$ and complete a table of three columns (n, x_n, y_n)

(ii) use a step of $h = 0.01$ and state only the approximation of $y(2)$ [6]

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